Seeking Nash Equilibria under Nonconvex Coupling Constraints

DEWS Colloquia – Università degli Studi dell'Aquila
IEEE CSS Italy Chapter Seminar Series

Paolo Scarabaggio

Department of Electrical and Information Engineering.
Polytechnic of Bari, Italy.

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Credit

Raffaele Carli
Assistant Professor
Politecnico di Bari

Sergio Grammatico
Associate Professor
TU Delft

Mariagrazia Dotoli
Full Professor
Politecnico di Bari

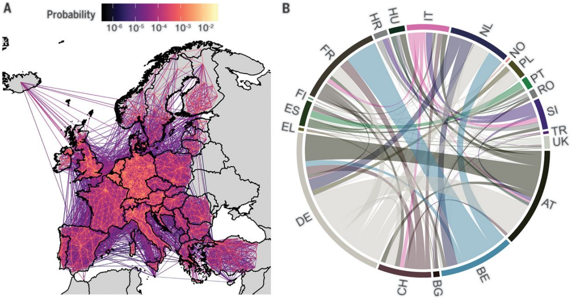
Introduction

Why do we need Game Theory?

COVID-19 pandemic: independent decision-makers (EU commission, governments, municipalities, institutions, citizens)

Uncoordinated reactions!

The selfish strategy of a decision-maker impacted all the others



From Ruktanonchai et al., Assessing the impact of coordinated COVID-19 exit strategies, Science, 2020

- **Players or decision-makers**: $i \in \mathcal{N} := \{1, \dots, N\} \subseteq \mathbb{N}$
- **Actions or strategies**: decision variable $\mathbf{x}_i \in \Omega_i \subseteq \mathbb{R}^n$

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_N \end{pmatrix} \in \mathbb{R}^{(N)n}, \qquad \forall i \in \mathcal{N} : \mathbf{x}_{-i} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{i-1} \\ \mathbf{x}_{i+1} \\ \vdots \\ \mathbf{x}_N \end{pmatrix} \in \mathbb{R}^{(N-1)n}$$

- **Outcomes or payoff functions**: $f_i\left(\mathbf{x}_i, \mathbf{x}_{-i}\right) : \mathbb{R}^n \times \mathbb{R}^{(N-1)n} \to \mathbb{R}$
  - Game: the "outcome" is dependent also on $\mathbf{x}_{-i}$, i.e., the strategies of the other players

Nash equilibrium problem

Several alternative solutions for the game: Nash equilibrium (NE)

Definition: Nash equilibrium

$$f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \inf \left\{ f_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \mid \mathbf{x}_i \in \Omega_i \right\}, \quad \forall i \in \mathcal{N}.$$

Nash equilibrium problem (NEP):

$$\forall i \in \mathcal{N} : \quad \begin{cases} \min\limits_{\mathbf{x}_i} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to} & \mathbf{x}_i \in \Omega_i \end{cases}$$

From a control-theoretic perspective, the objective is to develop a coordination mechanism for updating players' strategies to a NE.

Nash, Equilibrium points in N-person games, PNAS, 1950

Generalized Nash equilibrium problem

The strategies that a player can choose may be limited by the strategies of other players $\mathbf{x}_{-i}$, i.e., exists a coupled feasible set

$$\mathcal{X} = \Omega \cap \{\mathbf{x} \in \mathbb{R}^{Nn} \mid g(\mathbf{x}) \leq \mathbf{0}_M\}$$

hence $\mathbf{x}_i \in \mathcal{X}_i(\mathbf{x}_{-i})$

Definition: Generalized Nash equilibrium

$$f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \inf \{f_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \mid \mathbf{x}_i \in \mathcal{X}_i(\mathbf{x}_{-i}^*)\}, \forall i \in \mathcal{N}.$$

Generalized Nash equilibrium problem (GNEP):

$$\forall i \in \mathcal{N}: \quad \begin{cases} \min_{\mathbf{x}_i} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{s.t.} & \mathbf{x}_i \in \mathcal{X}_i(\mathbf{x}_{-i}). \end{cases}$$

Debreu, A social equilibrium existence theorem, PNAS, 1952

Minimum Principle

From an optimization point of view: a point $\mathbf{x}_i^*$ is said to be an optimal solution for a player if:

$$f(\mathbf{x}_i^*) \leq f(\mathbf{x}_i), \quad \forall \mathbf{x}_i \in \mathcal{X}_i(\mathbf{x}_{-i}^*)$$

when a point is an optimal solution? $\Rightarrow$ **optimality conditions**

Definition: Minimum principle for single valued functions

A feasible point $\mathbf{x}_i^*$ is an optimal solution if and only if

$$(\mathbf{y} - \mathbf{x}_i^*)^\top \nabla f(\mathbf{x}_i^*) \geq 0, \quad \forall \mathbf{y} \in \mathcal{X}_i(\mathbf{x}_{-i}^*)$$

equivalent to the Karush-Kuhn-Tucker (KKT) conditions when the set is defined by inequalities and equalities

Jointly convex GNE problem

How to prove the existence of a generalized Nash equilibrium?

...is quite difficult, we need some technical assumptions...

- **Continuity** of $f_i(\mathrm{x}_i, \mathrm{x}_{-i})$ and **compactness** of Ω_i
  - can be relaxed with further assumptions
- **Convexity** of $f_i(\mathrm{x}_i, \mathrm{x}_{-i})$, Ω_i and $\mathcal{X}$ required for fixed-point theorems (Brouwer, Kakutani, Banach-Picard, Caristi)

Jointly Convex Nash Equilibrium Problem

- Existence and uniqueness
- Convergence guarantee

sets $\mathcal{X}_i(\mathrm{x}_{-i})$ defined by a system of inequalities, componentwise convex with respect to all variables.

Facchinei, Fischer, Piccialli, On generalized Nash games and variational inequalities, ORL, 2007

Monotone Operators and Variational Inequality Theory

Definition: Variational inequality problem

The *variational inequality problem* $\mathrm{VI}(\mathcal{X}, F(\mathbf{x}))$ consists in finding $\mathbf{y} \in \mathcal{X}$ such that:

$$\inf_{\mathbf{x} \in \mathcal{X}} (\mathbf{x} - \mathbf{y})^\top F(\mathbf{y}) \geq 0.$$

$\mathcal{X} \subseteq \mathbb{R}^n$ closed and convex subset and $F : \mathcal{X} \to \mathbb{R}^n$ continuous map.

Definition: Quasi-variational inequality problem

The *quasi-variational inequality problem* $\mathrm{QVI}(\mathcal{X}(\mathbf{x}), F(\mathbf{x}))$ consists in finding $\mathbf{y} \in \mathcal{X}(\mathbf{x})$ such that:

$$\inf_{\mathbf{x} \in \mathcal{X}(\mathbf{x})} (\mathbf{x} - \mathbf{y})^\top F(\mathbf{y}) \geq 0.$$

$\mathcal{X}(\mathbf{x})$ point-to-set mapping and $F : \mathcal{X} \to \mathbb{R}^n$ continuous map.

(Quasi-)variational inequality problem

Solution set

- If F is monotone on $\mathcal{X}$, the solution set of VI, is closed and convex.

$$(F(\mathbf{x}) - F(\mathbf{y}))^\top (\mathbf{x} - \mathbf{y})) \geq 0, \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

- If F is strongly monotone on $\mathcal{X}$ with a constant $\mu \in \mathbb{R}_{>0}$, the solution set of VI, admits a unique solution.

$$(F(\mathbf{x}) - F(\mathbf{y}))^\top (\mathbf{x} - \mathbf{y})) \geq \mu \left\| \mathbf{x} - \mathbf{y} \right\|^2, \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

QVIs are much harder to solve!

We can solve some classes of (G)NEP by finding a solution for the associated *(quasi-)variational inequality* problem!

Variational reformulation of a game

VI associated with a NEP: $\text{VI}(\Omega, F(\mathbf{x}))$

$$F(\mathbf{x}) = \begin{bmatrix} \nabla_{\mathbf{x}_1} f_1(\mathbf{x}_1, \mathbf{x}_{-1}) \\ \vdots \\ \nabla_{\mathbf{x}_N} f_i(\mathbf{x}_N, \mathbf{x}_{-N}) \end{bmatrix}, \qquad \Omega = \prod_{i=1}^{N} \Omega_i$$

Solve the NEP $\Leftrightarrow$ Solve $\text{VI}(\Omega, F(\mathbf{x}))$

QVI associated with a GNEP: $\text{QVI}(\mathcal{X}, F(\mathbf{x}))$

$$\mathcal{X} = \Omega \cap \{\mathbf{x} \in \mathbb{R}^{Nn} \mid g(\mathbf{x}) \leq \mathbf{0}_M\} \quad \text{or} \quad \mathcal{X} = \prod_{i=1}^{N} \mathcal{X}_i(\mathbf{x}_{-i})$$

Solve the GNEP $\Leftrightarrow$ Solve $\text{QVI}(\mathcal{X}, F(\mathbf{x}))$

Jointly convex case:

Solve the GNEP $\Leftarrow$ Solve $\text{VI}(\mathcal{X}, F(\mathbf{x}))$

Variational reformulation of a game

Characterization of the solution obtained by the VI

$$\forall i \in \mathcal{N}: \quad \begin{aligned} &\nabla_{\mathrm{x}_i} f_i(\mathrm{x}_i, \mathrm{x}_{-i}) + \nabla_{\mathrm{x}_i} \phi(\mathrm{x}_i, \mathrm{x}_{-i}) \boldsymbol{\lambda}_i = \mathbf{0}_{Nn} \\ &\boldsymbol{\lambda}_i \geq \mathbf{0}_{Nn} \perp \phi(\mathrm{x}_i, \mathrm{x}_{-i}) \leq \mathbf{0}_{Nn} \end{aligned}$$

$$(\mathrm{x} - \mathrm{x}^*)^\top F(\mathrm{x}^*) \geq 0, \quad \forall \mathrm{x} \in \mathcal{X}.$$

KKT conditions of the corresponding VI$(\mathcal{X}, F(\mathrm{x}))$:

$$\mathrm{VI}: \quad \begin{aligned} &\begin{pmatrix} \nabla_{\mathrm{x}_1} f_1(\mathrm{x}_1, \mathrm{x}_{-1}) \\ \vdots \\ \nabla_{\mathrm{x}_i} f_i(\mathrm{x}_i, \mathrm{x}_{-i}) \end{pmatrix} + \begin{pmatrix} \nabla_{\mathrm{x}_1} \phi(\mathrm{x}) \\ \vdots \\ \nabla_{\mathrm{x}_i} \phi(\mathrm{x}) \end{pmatrix} \boldsymbol{\lambda} = 0 \\ &\boldsymbol{\lambda} \geq 0 \perp \phi(\mathrm{x}) \leq 0 \end{aligned}$$

Variational solutions: GNEP that are preserved passing to the VIP are exactly those for which all players have the same multipliers for the respective constraints: ”fair” behaviour

There is a convexity for the mapping F?

Monotonicity

- If f is convex $\Leftrightarrow$ F is monotone
- If f is strictly convex $\Leftrightarrow$ F is strictly monotone
- If f is strongly convex $\Leftrightarrow$ F is strongly monotone

Theorem: Existence of a GNE

Let us consider a jointly convex generalized game. A solution of the $\mathrm{VI}(\mathcal{X}, F(x))$ **exists**, is **unique** and is a solution of the original GNEP.

Harker, Generalized Nash games and quasi-variational inequalities. EJOR, 1991.

Facchinei, Kanzow. Generalized Nash equilibrium problems. AOR, 2007.

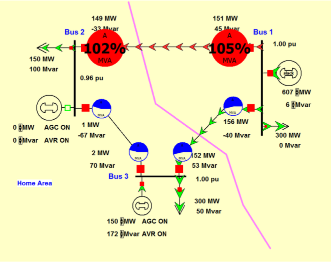
Motivating Examples

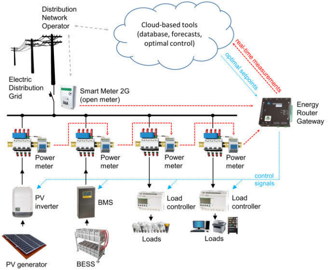
Noncooperative Optimal Power Flow in Power Grids

Nonconvex constraints: power flow equations

$$|V_b| \sum_{r \in \mathcal{B}} |V_r||Y_{br}| \cos(\theta_{br} + \theta_r - \theta_b) = P_b$$

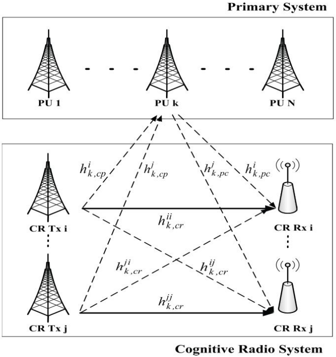
$$-|V_b| \sum_{r \in \mathcal{B}} |V_r||Y_{br}| \sin(\theta_{br} + \theta_r - \theta_b) = Q_b$$

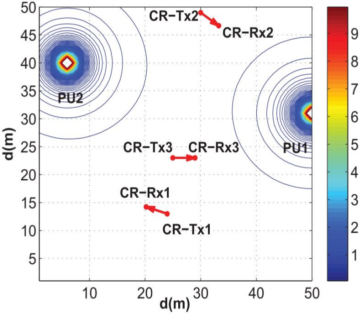




Power Allocation Games in Cognitive Radio Systems

Nonconvex constraints: power allocation in Gaussian frequency-selective interference channels

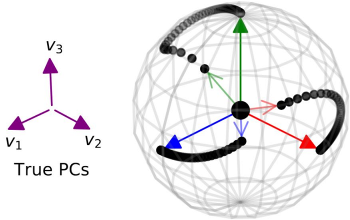




Eigengame: PCA as a Nash equilibrium

Principal component analysis (PCA) as a competitive game in which each approximate eigenvector is controlled by a player whose goal is to maximize their own utility function

parallelizing the computation by transforming the problem in a noncooperative game



Paolo Scarabaggio (paolo.scarabaggio@poliba.it)
Decision & Control Lab. (http://dclab.poliba.it/), Politecnico di Bari, Italy.

Clarke's Local Generalized Nash Equilibria

Due to the nature of several applications, the coupling feasible set $\mathcal{X}$ may result to be nonconvex.

Assumption 1: cost functions

For each $i \in \mathcal{N}$ and for every $\mathbf{x}_{-i}$, $f_i(\cdot, \mathbf{x}_{-i})$ convex and continuously differentiable.

Assumption 2: coupling feasible set

For each $m \in \mathcal{M}$ and for every $\mathbf{x}_{-i}$, $g_m(\cdot, \mathbf{x}_{-i})$ continuously differentiable (possibly nonconvex). $\mathcal{X}$ is nonempty and compact.

We analyze the particular case of nonconvex feasible sets... with nonconvex feasible sets.

Weaker equilibrium conditions:

- ▷ Clarke's local generalized Nash equilibrium (CL-GNE).

Clarke's tangent cone $\tilde{\mathcal{X}}(\mathbf{x}) := \mathbf{x} + T_{\mathrm{cl}}(\mathcal{X}, \mathbf{x})$ at a point $\mathbf{x}$ for the (nonconvex) set $\mathcal{X}$.

Definition: Clarke's local GNE

A CL-GNE is a collective strategy $\mathbf{x}^* \in \mathcal{X}$ such that for each $i \in \mathcal{N}$:

$$f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \inf \left\{ f_i(\mathbf{y}, \mathbf{x}_{-i}^*) \mid \mathbf{y} \in \tilde{\mathcal{X}}_i(\mathbf{x}_{-i}^*) \right\}$$

- ▷ If $\mathcal{X}$ is convex, the CL-GNE is equivalent to the GNE
- ▷ **Approximation of the set in a specific point.**
- ▷ Always convex even if $\mathcal{X}$ is nonconvex

Clarke's tangent cone

Clarke's tangent vector

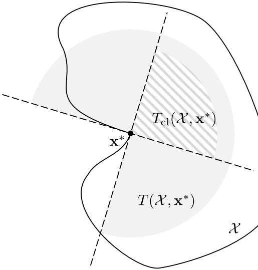
Nonempty subset $\mathcal{X}$ of a Euclidean space E, and a point $\mathbf{x} \in \mathrm{cl}(\mathcal{X})$.
$(\mathbf{x}_n)_{n\in\mathbb{N}} \in \mathcal{X}$, $(t_n)_{n\in\mathbb{N}}$ $(h_n)_{n\in\mathbb{N}} \in E$ such that:

$$\mathbf{x} = \lim_{n\to\infty} \mathbf{x}_n, \quad \lim_{n\to\infty} t_n = 0, \quad \mathbf{d} = \lim_{n\to\infty} d_n, \quad \mathbf{x}_n + t_n d_n \in \mathcal{X}$$

$\mathbf{d} \in E$ is a *Clarke's tangent vector* to $\mathcal{X}$ at $\mathbf{x}$.

Definition: Clarke's tangent cone

The set of all Clarke's tangent vectors to $\mathcal{X}$ at $\mathbf{x}$ is the *Clarke's tangent cone* $T_{\mathrm{cl}}(\mathcal{X}, \mathbf{x})$.



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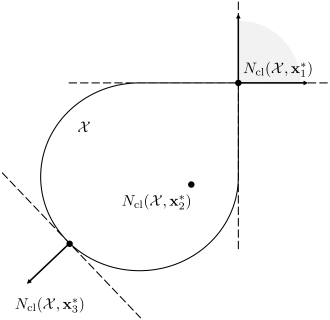
Clarke's local generalized Nash equilibrium

For each player $i \in \mathcal{N}$:

- ▷ optimality condition
  $-\nabla f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \in N_{\mathrm{cl}}(\mathcal{X}_i(\mathbf{x}_{-i}^*), \mathbf{x}_i^*)$
- ▷ *Karush–Kuhn–Tucker* (KKT) conditions

$$\mathrm{KKT}_i : \begin{cases} 0 \in \nabla_{\mathbf{x}_i} f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) + \nabla_{\mathbf{x}_i} g(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)\boldsymbol{\lambda}_i \\ \mathbf{0}_M \leq \boldsymbol{\lambda}_i \perp g(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \mathbf{0}_M \end{cases}$$



Theorem: Characterization

If $\mathbf{x}^* \in \mathcal{X}$ is a CL-GNE, we have that for each $i \in \mathcal{N}$:

(i) $-\nabla f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \in N_{\mathrm{cl}}(\mathcal{X}_i(\mathbf{x}_{-i}^*), \mathbf{x}_i^*)$;

(ii) there exists a vector $\boldsymbol{\lambda}_i \in \mathbb{R}^M_{\geq 0}$ satisfying the KKT conditions.

Variational Clarke's local GNE

We define a *quasi-variational inequality problem* QVI($\tilde{\mathcal{X}}(\mathbf{x}), F(\mathbf{x})$)

$$\inf_{\mathbf{y}\in\tilde{\mathcal{X}}(\mathbf{x}^*)} (\mathbf{y}-\mathbf{x}^*)^\top F(\mathbf{x}^*) \geq 0.$$

We define the *variational Clarke local generalized Nash equilibrium* (vCL-GNE) as a solution of the CL-GNEP that satisfies the QVIP.

Similarly to the relation between GNEP and VIP, not all the solutions of the CL-GNEP are solutions of the QVIP; viceversa, a solution of the QVIP is a solution of the original CL-GNEP.

- **Solve the GNEP $\Leftrightarrow$ Solve QVI($\mathcal{X}, F(\mathbf{x})$)**
- **Solve the GNEP $\Leftarrow$ Solve VI($\mathcal{X}, F(\mathbf{x})$)** (Jointly convex case)
- **Solve the CL-GNEP $\Leftarrow$ Solve QVI($\tilde{\mathcal{X}}(\mathbf{x}), F(\mathbf{x})$)**

$\tilde{\mathcal{X}}(\mathbf{x})$ is convex: only variational solutions!

At a vCL-GNE, we have that in a local subset of $\mathcal{X}$ each agent cannot unilaterally maximize their own function while keeping the strategies of the other agents fixed (locally fair equilibrium point).

Theorem: Characterization

(i) Let $\mathbf{x}^*$ be a solution of the CL-GNEP, where the KKT conditions for all players hold with the same Lagrangian multipliers $\boldsymbol{\lambda} = \boldsymbol{\lambda}_i, \forall i \in \mathcal{N}$. Then, $\mathbf{x}^*$ is a solution of the QVI and thus it is a vCL-GNE.

(ii) Viceversa, let $\mathbf{x}^*$ be a solution of the QVI and thus be a vCL-GNE. Then, $\mathbf{x}^*$ is a solution of the CL-GNEP at which the KKT conditions hold with the same Lagrangian multipliers, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_i, \forall i \in \mathcal{N}$.

Example 1

Two players, with strategies $\mathbf{x}_1 \in \mathbb{R}$ and $\mathbf{x}_2 \in \mathbb{R}$

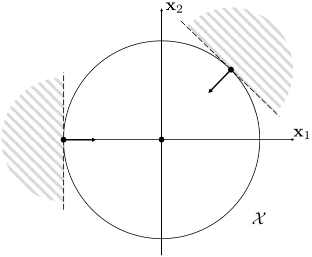
$$\forall i \in \{1,2\} : \begin{cases} \min\limits_{-2 \le \mathbf{x}_i \le 2} & f_i(\mathbf{x}_i) = \mathbf{x}_i^2 \\ \text{s.t.} & \mathbf{x}_1^2 + \mathbf{x}_2^2 \ge 1. \end{cases}$$

Cost functions decoupled and strictly convex.

The game has infinite CL-GNE

$$\left.\begin{aligned} 2\mathbf{x}_i - \lambda_i 2\mathbf{x}_i &= 0 \\ \lambda_i \ge 0 \perp \mathbf{x}_1^2 + \mathbf{x}_2^2 &\ge 1 \end{aligned}\right. \quad \forall i = 1,2.$$

All points on the unitary circumference $\lambda_1 = \lambda_2 = 1$; hence, all these points are also vCL-GNE.



Example 2

Modified version of Example 1.

$$\forall i \in \{1,2\} : \begin{cases} \min\limits_{-2 \le x_i \le 2} & f_i(x_i) = x_i^2 \\ \text{s.t.} & (x_1 - \frac{1}{4})^2 + (x_2 - \frac{1}{4})^2 \ge 1. \end{cases}$$

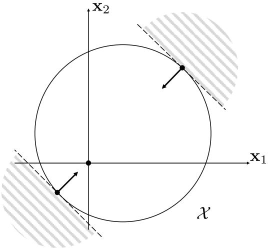
$\forall i = 1, 2$

$$2x_i - \lambda_i(2x_i - \tfrac{1}{2}) = 0$$

$$\lambda_i \ge 0 \perp 1 - (x_1 - \tfrac{1}{4})^2 - (x_2 - \tfrac{1}{4})^2 \le 0$$

Only two vCL-GNE

- $x_1 = x_2 = \frac{1}{4} + \frac{\sqrt{2}}{2}$
- $x_1 = x_2 = \frac{1}{4} - \frac{\sqrt{2}}{2}$



Existence and Uniqueness

Existence and local uniqueness

- Projection operator

$$\mathrm{proj}_{\mathcal{X}}(\mathsf{y}) := \operatorname{argmin}_{\mathsf{x}\in\mathcal{X}} \|\mathsf{x}-\mathsf{y}\|$$

- Distance operator between a point and a set

$$\mathrm{dist}(\mathsf{y},\mathcal{X}) := \min_{\mathsf{x}\in\mathcal{X}} \|\mathsf{x}-\mathsf{y}\|$$

$F(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with a constant $\ell \in \mathbb{R}_{>0}$ if

$$\|F(\mathsf{x}) - F(\mathsf{y})\| \leq \ell \|\mathsf{x}-\mathsf{y}\|, \ \forall \mathsf{x},\mathsf{y} \in \mathbb{R}^n$$

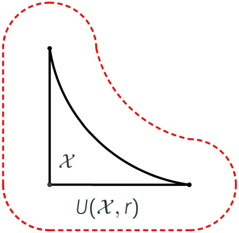
- for $\ell = 1$, we say F is nonexpansive
- for $\ell < 1$, we say F is a contraction (with contraction factor ℓ)

If $\mathcal{X}$ is convex $\mathrm{proj}_{\mathcal{X}}(\mathsf{y})$ is a contraction: existence and convergence proofs based on projected gradient $\mathsf{x}^* \in \mathrm{fix}(\mathrm{proj}_{\mathcal{X}}(\mathrm{Id} - \gamma F(\cdot)))$

Proximal smoothness

The projection onto nonconvex sets is not a nonexpansive operator.

- Can we ensure that $\mathrm{proj}_{\mathcal{X}}(\mathbf{y})$ is a contraction when $\mathcal{X}$ is nonconvex?
- Can we still work with a nonexpansive operator (or expansive with a certain degree)?



For any set $\mathcal{X}$ and $r > 0$, one sets
$U(\mathcal{X}, r) := \{\mathbf{r} \in \mathbb{R}^n | \mathrm{dist}(\mathbf{y}, \mathcal{X}) < r\}$

Definition: Proximally Smooth Sets

A set $\mathcal{X} \subseteq \mathbb{R}^n$ is said to be proximally smooth if there exists $r > 0$ such that the distance function $\mathrm{dist}(\cdot, \mathcal{X})$ is continuously differentiable on the r-enlargement $U(\mathcal{X}, r) := \mathcal{X} + r\mathbb{B}$.

Federer, Curvature measures. TAMS, 1959.

Paolo Scarabaggio (paolo.scarabaggio@poliba.it)
Decision & Control Lab. (http://dclab.poliba.it/), Politecnico di Bari, Italy.

Proximal smoothness

Proximal smoothness properties

Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a nonempty closed set. If $\mathcal{X}$ is r-proximally smooth, then the following properties hold for any $r' \in (0, r)$:

(i) $\mathrm{proj}_{\mathcal{X}}(\mathbf{x}) \neq \varnothing$ for all $\mathbf{x} \in U(\mathcal{X}, r')$;

(ii) $\mathrm{proj}_{\mathcal{X}}(\mathbf{x})$ is a singleton for all $\mathbf{x} \in U(\mathcal{X}, r')$;

(iii) $\mathrm{proj}_{\mathcal{X}}(\cdot)$ is p-Lipschitz continuous on $U(\mathcal{X}, r')$, where $p = {}^{r}/_{(r-r')}$;

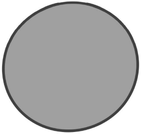
(iv) $N_{\mathrm{px}}(\mathcal{X}, \cdot)$ is closed as a set-valued mapping;

$\mathrm{proj}_{\mathcal{X}}(\mathbf{x}) : U(\mathcal{X}, r') \to \mathcal{X}$ **well-defined and norm-to-norm continuous**.

Informally speaking: local nonconvexities counterbalanced with smoothness of constraints

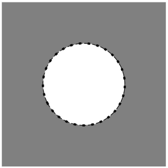
Federer, Curvature measures. TAMS, 1959.

Proximal smoothness

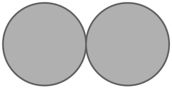


Nonempty closed convex set: ∞-proximal smooth

Lack of proximal smoothness ("angle")



$\mathbb{R}^n \setminus B(0, r)$ is r-proximal smooth



Non-proximal smooth union of two convex sets

Paolo Scarabaggio (paolo.scarabaggio@poliba.it)
Decision & Control Lab. (http://dclab.poliba.it/), Politecnico di Bari, Italy.

Preservation of proximal smoothness

Sufficient conditions guaranteeing the prox-regularity for:

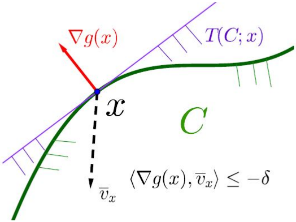
- A set defined by inequality/equality constraints
- Intersection of two prox-regular sets

Differentiable and Lipschitz cont. inequality/equality on $U(\mathcal{X}, r)$

(S.U.) $\exists \delta > 0, \forall x \in \mathrm{bd} C, \exists \overline{v}_x \in \mathbb{B}, \forall k \in \{1, \ldots, m\}, \forall \xi \in \partial_C g_k(x), \langle \xi, \overline{v}_x \rangle \leq -\delta$

Single coupling constraint:
continuous differentiability only!

Adly, Thibault, Preservation of Prox-Regularity of Sets with Applications to Constrained Optimization. SIAM JO, 2016.



Existence and local uniqueness

Assumptions 1 and 2 hold

Proposition: Existence

$\mathcal{X}$ r-proximally smooth, for any $\gamma \in (0, \frac{r'}{1+\bar{F}_{U(\mathcal{X},r)}})$ with $r' \in (0, r)$:

(i) $\mathbf{x}^* \in \mathcal{X}$ is a vCL-GNE;

(ii) $\mathbf{x}^* = \mathrm{proj}_{\mathcal{X}}(\mathbf{x}^* - \gamma F(\mathbf{x}^*))$, i.e., $\mathbf{x}^* \in \mathrm{fix}(\mathrm{proj}_{\mathcal{X}}(\mathrm{Id} - \gamma F(\cdot)))$.

then the CL-GNEP has at least one vCL-GNE.

An additional assumption on the mapping F is required

Proposition: Local uniqueness

Mapping F is strictly monotone, then any vCL-GNE $\mathbf{x}^* \in \mathcal{X}$ is unique in its Clarke's tangent cone $\tilde{\mathcal{X}}(\mathbf{x}^*)$.

Equilibrium Computation

Strongly monotone pseudo-gradient mapping

Assumption 3: strongly monotone pseudo-gradient mappings

$\mathcal{X}$ is r-proximally smooth. Pseudo-gradient mapping F strongly monotone with $\mu > 0$ and Lipschitz continuous with $\ell > 0$.

$$(F(\mathrm{x}) - F(\mathrm{y}))^\top (\mathrm{x} - \mathrm{y})) \geq \mu \left\| \mathrm{x} - \mathrm{y} \right\|^2, \ \forall \mathrm{x}, \mathrm{y} \in \mathbb{R}^n$$

Algorithm 1: classical projected approach

$$\mathrm{x}^{k+1} = \mathrm{proj}_{\mathcal{X}}(\mathrm{x}^k - \gamma F(\mathrm{x}^k))$$

Theorem: Existence and convergence

Let $\gamma \in (0, \min\{\frac{\sqrt{\ell^2+p^2(\mu^2-\ell^2)}+\mu p}{\ell^2 p}, \frac{r'}{1+\bar{F}_{U(\mathcal{X},r)}}\}), \mu \geq \ell$ and $r' \in (0, r)$.

(i) The CL-GNEP has at least one vCL-GNE;

(ii) The sequence generated by Algorithm 1 converges to a vCL-GNE.

Strongly monotone pseudo-gradient mapping

To preserve the smoothness properties $\mathbf{x}^k - \gamma F(\mathbf{x}^k) \in U(\mathcal{X}, r')$

$$\left\|\mathbf{x}^{k+1} - \mathbf{x}^*\right\|^2 = \|\mathrm{proj}_{\mathcal{X}}(\mathbf{x} - \gamma F(\mathbf{x})) - \mathrm{proj}_{\mathcal{X}}(\mathbf{x}^* - \gamma F(\mathbf{x}^*))\|^2$$

$$\leq p^2 \left\|(\mathbf{x}^k - \gamma F(\mathbf{x}^k)) - (\mathbf{x}^* - \gamma F(\mathbf{x}^*))\right\|^2$$

$$= p^2 \left\|(\mathbf{x}^k - \mathbf{x}^*) - \gamma(F(\mathbf{x}^k) - F(\mathbf{x}^*))\right\|^2$$

$$= p^2 (\left\|\mathbf{x}^k - \mathbf{x}^*\right\|^2 + \gamma^2 \left\|F(\mathbf{x}^k) - F(\mathbf{x}^*)\right\|^2 - 2\gamma(F(\mathbf{x}^k) - F(\mathbf{x}^*))^\top (\mathbf{x}^k - \mathbf{x}^*))$$

$$\leq p^2 (1 - 2\gamma\mu + \gamma^2 \ell^2) \left\|\mathbf{x}^k - \mathbf{x}^*\right\|^2$$

For proximally smooth sets the projection is Lipschitz continuous with constant $p = {}^{r}/_{r-r'}$.

Expansiveness of projection counterbalanced with the strong monotonicity of the pseudo-gradient mapping

Monotone pseudo-gradient mapping

Assumption 4: monotone pseudo-gradient mappings

$\mathcal{X}$ is r-proximally smooth. F is monotone and Lipschitz continuous with constant $\ell > 0$.

$$(F(\mathbf{x}) - F(\mathbf{y}))^\top (\mathbf{x} - \mathbf{y})) \geq 0, \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

Algorithm 2: Modified version of the Korpelevich's approach.

$$\mathbf{x}^{k+1} = \mathrm{proj}_{\tilde{\mathcal{X}}(\mathbf{y}^k)}(\mathbf{x}^k - \gamma F(\mathrm{proj}_{\mathcal{X}}(\mathbf{x}^k - \gamma F(\mathbf{x}^k)))$$

Theorem: Existence and convergence

Let $\gamma \in (0, \min\{\frac{1}{\ell}, \frac{r}{4(1+\bar{F}_{U(\mathcal{X},r)})}\})$ with $r' \in (0, r)$.

(i) The CL-GNEP has at least one vCL-GNE;

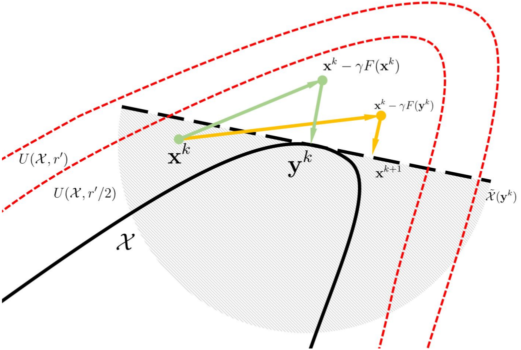
(ii) The sequence generated by Algorithm 2 converges to a vCL-GNE.

Monotone pseudo-gradient mapping

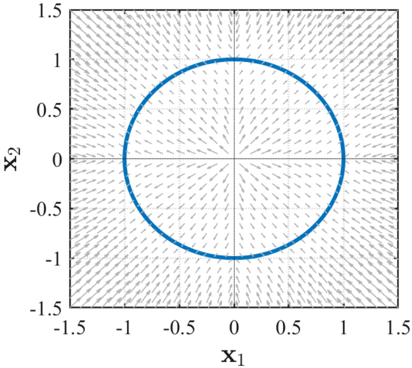
$$\begin{cases} \mathbf{y}^k = \mathrm{proj}_{\mathcal{X}}(\mathbf{x}^k - \gamma F(\mathbf{x}^k)) \\ \mathbf{x}^{k+1} = \mathrm{proj}_{\tilde{\mathcal{X}}(\mathbf{y}^k)}(\mathbf{x}^k - \gamma F(\mathbf{y}^k)) \end{cases}$$

Each iteration $\in U(\mathcal{X}, r')$.

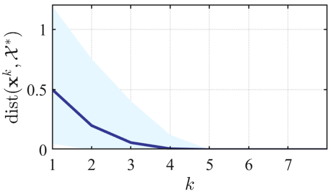
- $\mathbf{y}^k \in \mathcal{X}$
- $\mathbf{x}^k \in U(\mathcal{X}, r'/2)$.
- $\mathbf{x}^k - \gamma F(\mathbf{x}^k) \in U(\mathcal{X}, r')$.



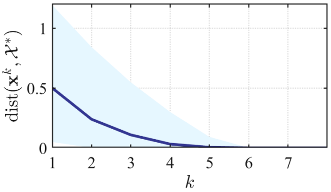
Example 1



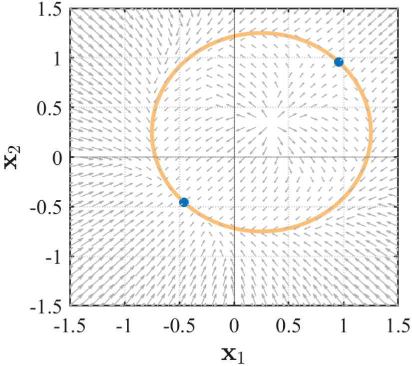
Algorithm 1



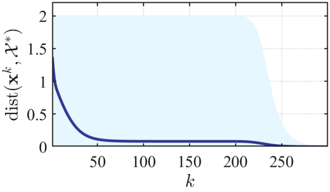
Algorithm 2



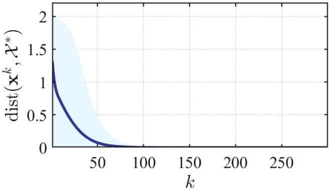
Example 2



Algorithm 1



Algorithm 2



Conclusion

Conclusions and Research Outlooks

- Definition of the problem ✓
- Optimality of the solution ✓
- Existence for proximally smooth sets ✓
- Uniqueness for proximally smooth sets ✓
- Convergence to a vCL-GNE on (strongly) monotone operators ✓
- Existence on more general settings ×
- Distributed convergence ×
- And so on...

Scarabaggio, Carli, Grammatico & Dotoli. **Clarke's Local Equilibria in Nash Games with Nonconvex Coupling Constraints**. DOI: 10.36227/techrxiv.20079959

Scarabaggio, Carli & Dotoli. **Noncooperative Equilibrium Seeking in Distributed Energy Systems Under AC Power Flow Nonlinear Constraints.** *IEEE Transactions on Control of Network Systems, 2022*

Thanks for your attention!

Questions?